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Multi-gluon collinear limits from MHV diagrams

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Abstract: We consider the multi-collinear limit of multi-gluon QCD amplitudes at tree level. We use the MHV rules for constructing colour ordered tree amplitudes and the general collinear factorization formula to derive timelike splitting functions that are valid for specific numbers of negative helicity gluons and an arbitrary number of positive helicity gluons (or vice versa). As an example we present new results describing the collinear limits of up to six gluons.

Keywords: Duality in Gauge Field Theories, NLO Computations, QCD.

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Contents

1. Introduction

The interpretation of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and QCD as a topological string propagating in twistor space [1], has inspired a new and powerful framework for computing tree-level and one-loop scattering amplitudes in Yang-Mills gauge theory. Notably, two distinct formalisms have been developed for calculations of scattering amplitudes in gauge theory — the 'MHV rules' of Cachazo, Svrček and Witten (CSW) [2], and the 'BCF recursion relations' of Britto, Cachazo, Feng and Witten [3, 4].

In this paper, we wish to exploit these formalisms to examine the singularity structure of tree-level amplitudes when many gluons are simultaneously collinear. Understanding the infrared singular behaviour of multi-parton amplitudes is a prerequisite for computing infrared-finite cross sections at fixed order in perturbation theory. In general, when one or more final state particles are either soft or collinear, the amplitudes factorise. The first factor in this product is an amplitude depending on the remaining hard partons in the process (including any hard partons constructed from an ensemble of unresolved partons). The second factor contains all of the singularities due to the unresolved particles. One of the best known examples of this type of factorisation is the limit of tree amplitudes when two particles are collinear. This factorisation is universal and can be generalised to any number of loops [5].

Both frameworks, the MHV rules and the BCF recursion relations, are remarkably powerful in deriving analytic expressions for massless multi-particle tree-level amplitudes. At the same time, for the specific purpose of deriving general multi-collinear limits, we find the MHV rules approach to be particularly convenient.

A useful feature of the MHV rules is that it is not required to set reference spinors η_α and $\eta_{\dot\alpha}$ to specific values dictated by kinematics or other reasons. In this way, on-shell (gauge-invariant) amplitudes are derived for arbitrary η's, i.e. without fixing the gauge. By starting from the appropriate colour ordered amplitude and taking the collinear limit, the full amplitude factorises into an MHV vertex multiplied by a multi-collinear splitting function that depends on the helicities of the collinear gluons. Because the MHV vertex is a single factor, the collinear splitting functions have a similar structure to MHV amplitudes. Furthermore, the gauge or η-dependence of the splitting function drops out.

One of the main points of our approach is that, in order to derive all required splitting functions we do not need to know the full amplitude. Out of the full set of MHV-diagrams contributing to the full amplitude, only a subset will contribute to the multi-collinear limit. This subset includes only those MHV-diagrams which contain an internal propagator which goes on-shell in the multi-collinear limit. In other words, the IR singularities in the MHV approach arise entirely from internal propagators going on-shell. This observation is specific to the MHV rules method and does not apply to the BCF recursive approach. We will see in section 4.2.3 that in the BCF picture collinear splitting functions generically receive contributions from the full set of allowed BCF diagrams.[1] In view of this, we will employ the MHV rules of [2] for setting up the formalism and for derivations of general multi-collinear amplitudes. At the same time, various specific examples of multi-collinear splitting amplitudes derived in this paper will also be checked in section 4.2 using the BCF recursion relations [3].

The basic building blocks of the MHV rules approach [2] are the colour-ordered n-point vertices which are connected by scalar propagators. These MHV vertices are off-shell continuations of the maximally helicity-violating (MHV) n-gluon scattering amplitudes of Parke and Taylor [6, 7]. They contain precisely two negative helicity gluons. Written in terms of spinor inner products [8], they are composed entirely of the holomorphic products $\langle i\, j\rangle$ of the right-handed (undotted) spinors, rather than their anti-holomorphic partners $[i\, j]$,

$$A_n(1^+, \ldots, p^-, \ldots, q^-, \ldots, n^+) = \frac{\langle p\, q\rangle^4}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle n-1,\, n\rangle\, \langle n\, 1\rangle}\,, \tag{1.1}$$

where we introduce the common notation $\langle p_i\, p_j\rangle = \langle i\, j\rangle$ and $[p_i\, p_j] = [i\, j]$. By connecting MHV vertices, amplitudes involving more negative helicity gluons can be built up.

[1]This is because the required IR poles in the BCF approach arise not only from propagators going on-shell, but also from the constituent BCF vertices.

The MHV rules for gluons [2] have been extended to amplitudes with fermions [9]. New compact results for tree-level gauge-theory results for non-MHV amplitudes involving arbitrary numbers of gluons [10 – 12], and fermions [9, 13 – 15] have been derived. They have been applied to processes involving external Higgs bosons [16, 17] and electroweak bosons [18]. MHV rules for tree amplitudes have further been recast in the form of recursive relations [12, 18, 17] which facilitate calculations of higher order non-MHV amplitudes in terms of the known lower-order results. In many cases new classes of tree amplitudes were derived, and in all cases, numerical agreement with previously known amplitudes has been found.

MHV rules have also been shown to work at the loop-level for supersymmetric theories. Building on the earlier work of Bern et al [19, 20], there has been enormous progress in computing cut-constructible multi-leg loop amplitudes in $\mathcal{N} = 4$ [21 – 30] and $\mathcal{N} = 1$ [31, 33, 32, 34] supersymmetric gauge theories. Encouraging progress has also been made for non-supersymmetric loop amplitudes [35 – 37].

Remarkably, the expressions obtained for the infrared singular parts of $\mathcal{N} = 4$ one-loop amplitudes (which are known to be proportional to tree-level results) were found to produce even more compact expressions for gluonic tree amplitudes [30, 38]. This observation led to the BCF recursion relations [3, 4] discussed earlier as well as extremely compact six-parton amplitudes [39, 40] and expressions for MHV and NMHV graviton amplitudes [41, 42].

The factorisation properties of amplitudes in the infrared play several roles in developing higher order perturbative predictions for observable quantities. First, a detailed knowledge of the structure of unresolved emission enables phase space integrations to be organised such that the infrared singularities due to soft or collinear emission can be analytically extracted [45 – 47]. Second, they enable large logarithmic corrections to be identified and resummed. Third, the collinear limit plays a crucial role in the unitarity-based method for loop calculations [19, 20, 48, 49].

In general, to compute a cross section at N^{n}LO, one requires detailed knowledge of the infrared factorisation functions describing the unresolved configurations for n-particles at tree-level, $(n-1)$-particles at one-loop etc. The universal behaviour in the double collinear limit is well known at tree-level (see for example refs. [50, 51]), one-loop [52, 19, 53 – 56] and at two-loops [57, 58]. Similarly, the triple collinear limit has been studied at tree-level [59 – 62] and, in the case of distinct quarks, at one-loop [63]. Finally, the tree-level quadruple gluon collinear limit is derived in ref. [64].

Our paper is organised as follows. In section 2, we briefly review the spinor helicity and colour ordered formalism that underpins the MHV rules. Section 3 describes the procedure for taking the collinear limit and deriving the splitting functions. We write down a general collinear factorization formula, which is valid for specific numbers of negative helicity gluons and an arbitrary number of positive helicity gluons and demonstrate that the gauge dependence explicitly cancels. We find it useful to classify our results according to the difference between the number of negative helicity gluons before taking the collinear limit, and the number after. We call this difference ΔM. We provide formulae describing an arbitrary number of gluons for $\Delta M \leq 2$ in section 4.1. Specific explicit results for the collinear limits of up to six gluons are given in section 4.42. We have numerically checked that our results agree with the results available in the literature for three and four collinear gluons [64]. Our findings are summarized in section 5.

2. Colour ordered amplitudes in the spinor helicity formalism

Tree-level multi-gluon amplitudes can be decomposed into colour-ordered partial amplitudes as

$$\mathcal{A}_n(\{p_i, \lambda_i, a_i\}) = ig^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}})\, A_n(\sigma(1^{\lambda_1}, \ldots, n^{\lambda_n}))\,. \tag{2.1}$$

Here S_n/Z_n is the group of non-cyclic permutations on n symbols, and j^{λ_j} labels the momentum p_j and helicity λ_j of the j^{th} gluon, which carries the adjoint representation index a_i. The T^{a_i} are fundamental representation $\mathrm{SU}(N_c)$ colour matrices, normalized so that $\mathrm{Tr}(T^a T^b) = \delta^{ab}$. The strong coupling constant is $\alpha_s = g^2/(4\pi)$. The MHV rules method of ref. [2] is used to evaluate only the purely kinematic amplitudes A_n. Full amplitudes are then determined uniquely from the kinematic part A_n, and the known expressions for the colour traces.

In the spinor helicity formalism [8, 6, 7] an on-shell momentum of a massless particle, $p_\mu p^\mu = 0$, is represented as

$$p_{a\dot{a}} \equiv\ p_\mu \sigma^\mu_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}\,, \tag{2.2}$$

where λ_a and $\tilde{\lambda}_{\dot{a}}$ are two commuting spinors of positive and negative chirality. Spinor inner products are defined by[2]

$$\langle \lambda, \lambda' \rangle = \epsilon_{ab} \lambda^a \lambda'^b\,, \qquad [\tilde{\lambda}, \tilde{\lambda}'] = -\epsilon_{\dot{a}\dot{b}} \tilde{\lambda}^{\dot{a}} \tilde{\lambda}'^{\dot{b}}\,, \tag{2.3}$$

and a scalar product of two null vectors, $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ and $q_{a\dot{a}} = \lambda'_a \tilde{\lambda}'_{\dot{a}}$, becomes

$$p_\mu q^\mu = -\frac{1}{2} \langle \lambda, \lambda' \rangle [\tilde{\lambda}, \tilde{\lambda}']\,. \tag{2.4}$$

The MHV rules of ref. [2] were developed for calculating purely gluonic amplitudes at tree level. In this approach all non-MHV n-gluon amplitudes (including $\overline{\mathrm{MHV}}$) are expressed as sums of tree diagrams in an effective scalar perturbation theory. The vertices in this theory are the MHV amplitudes of eq. (1.1) continued off-shell as described below, and connected by scalar propagators $1/q^2$.

When one leg of an MHV vertex is connected by a propagator to a leg of another MHV vertex, both legs become internal to the diagram and have to be continued off-shell. Off-shell continuation is defined as follows [2]: we pick an arbitrary reference spinor $\eta^{\dot{a}}$ and define λ_a for any internal line carrying momentum $q_{a\dot{a}}$ by

$$\lambda_a = q_{a\dot{a}} \eta^{\dot{a}}\,. \tag{2.5}$$

External lines in a diagram remain on-shell, and for them λ is defined in the usual way. For the off-shell lines, the same reference spinor η is used in all diagrams contributing to a given amplitude.

[2]Our conventions for spinor helicities follow [1, 2], except that $[ij] = -[ij]_{CSW}$ as in ref. [65].

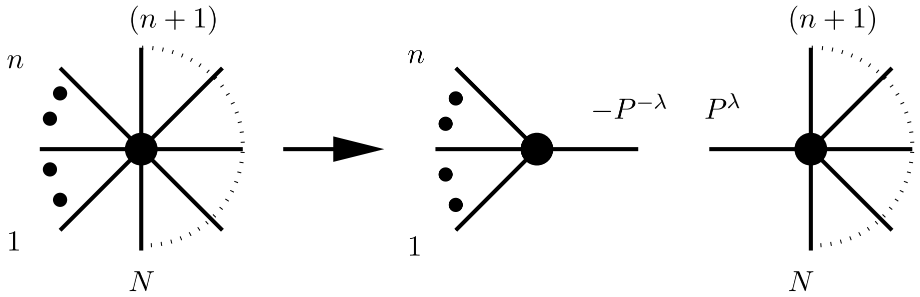

Figure 1: Factorisation of an N-point colour ordered amplitude with gluons $p_1, \ldots, p_n$ collinear into splitting function for $P \to 1, \ldots, n$ multiplied by an $(N-n+1)$-point amplitude.

3. The multiple collinear limit

To find the splitting functions we work with the colour stripped amplitudes A_n. For these colour ordered amplitudes, it is known that when the collinear particles are not adjacent there is no collinear divergence [64]. Therefore, without loss of generality, we can take particles $1 \ldots n$ collinear.

The multiple collinear limit is approached when the momenta $p_1, \ldots, p_n$ become parallel. This implies that all the particle subenergies $s_{ij} = (p_i + p_j)^2$, with $i, j = 1, \ldots, n$, are simultaneously small. We thus introduce a pair of light-like momenta P^ν and ξ^ν $(P^2 = 0, \xi^2 = 0)$, and we write

$$(p_1 + \cdots + p_n)^\nu = P^\nu + \frac{s_{1,n}\, \xi^\nu}{2\, \xi \cdot P}\,, \qquad s_{i,j} = (p_i + \cdots + p_j)^2\,, \tag{3.1}$$

where $s_{1,n}$ is the total invariant mass of the system of collinear partons. In the collinear limit, the vector P^ν denotes the collinear direction, and the individual collinear momenta are $p_i^\nu \to z_i P^\nu$. Here the longitudinal-momentum fractions z_i are given by

$$z_i = \frac{\xi \cdot p_i}{\xi \cdot P} \tag{3.2}$$

and fulfil the constraint $\sum_{i=1}^{m} z_i = 1$. To be definite, in the rest of the paper we work in the time-like region so that $(s_{ij} > 0,\ 1 > z_i > 0)$.

As illustrated in figure 1, in the multi-collinear limit an N-gluon colour ordered tree amplitude factorises and can be written as

$$A_N(1^{\lambda_1}, \ldots, N^{\lambda_N}) \to \mathrm{split}(1^{\lambda_1}, \ldots, n^{\lambda_n} \to P^\lambda) \times A_{N-n+1}((n+1)^{\lambda_{n+1}}, \ldots, N^{\lambda_N}, P^\lambda)\,. \tag{3.3}$$

This labelling of the splitting amplitude $\mathrm{split}(1^{\lambda_1}, \ldots, n^{\lambda_n} \to P^\lambda)$ differs from the usual definition because we use the momentum and helicity that participates in the resultant amplitude P^λ rather than $-P^{-\lambda}$. With this choice, it is easier to see how the helicity is conserved in the splitting, i.e. helicity $\lambda^1, \ldots, \lambda^n$ is replaced by λ.

There are two different types of collinear limit [2], those that conserve the number of negative helicity gluons between the initial state and the final collinear state, and those that do not.

Only the limits of the type split$(1^+, \ldots, n^+ \to P^+)$ and split$(1^-, 2^+, \ldots, n^+ \to P^-)$ can contribute to the negative helicity conserving case, and these collinear splitting functions are straightforward to derive directly from the simple MHV vertex.

All other limits belong to the second class which do not conserve the number of negative helicity gluons, and therefore we classify our results according to the difference between the number of negative helicity gluons before taking the collinear limit, and the number after, ΔM. We find that ΔM corresponds to the order of MHV diagram needed to find a particular collinear limit, as follows,

$$\begin{aligned}
\Delta M = 0 \Rightarrow \qquad & \text{MHV}: \; 1^+, 2^+, 3^+, \ldots, n^+ \to P^+ \\
& \phantom{\text{MHV}: \;} 1^-, 2^+, 3^+, \ldots, n^+ \to P^- \\
\\
\Delta M = 1 \Rightarrow \qquad & \text{NMHV}: \; 1^-, 2^+, 3^+, \ldots, n^+ \to P^+ \\
& \phantom{\text{NMHV}: \;} 1^-, 2^-, 3^+, \ldots, n^+ \to P^- \\
\\
\Delta M = 2 \Rightarrow \qquad & \text{NNMHV}: \; 1^-, 2^-, 3^+, \ldots, n^+ \to P^+ \\
& \phantom{\text{NNMHV}: \;} 1^-, 2^-, 3^-, \ldots, n^+ \to P^-
\end{aligned} \tag{3.4}$$

and so on for all $\Delta M > 2$ cases.

The splitting functions are derived by examining the general form of MHV diagrams, which consist of MHV vertices and scalar propagators. The general form of the n-particle collinear splitting functions is given by

$$\Delta M = 0: \quad \propto \frac{1}{\langle \ \rangle^{n-1}}, \tag{3.5}$$

$$\Delta M \neq 0: \quad \propto \frac{1}{[\ ]^{v-1} \langle \ \rangle^{n-v}}, \tag{3.6}$$

such that $(v-1) + (n-v) = n-1$, where v is the number of vertices, and thus $v-1$ is the number of scalar propagators. From (3.6) it follows that for an MHV-diagram to contribute to $\Delta M \neq 0$ collinear limits, it is required to contain anti-holomorphic spinor products $[i\,j]$ of collinear momenta. However, because on-shell MHV vertices are entirely holomorphic, within the MHV rules there are only two potential sources of the anti-holomorphic spinor products. One source is scalar propagators $1/s_{ij} = 1/\langle i\,j\rangle\,[j\,i]$ which inter-connect MHV vertices. The second source is the off-shell continuation of the corresponding connected legs in the MHV vertices. Each off-shell continued leg of momentum P gives rise to a factor $\langle iP\rangle \propto \langle i|P|\eta]$ which gives rise to anti-holomorphic factors of $[j\eta]$. When the reference spinors $\eta_{\dot\alpha}$ are kept general, and specifically, not set to be equal to one of the momenta in the collinear set, the η-dependence must cancel and the off-shell continuation cannot give rise to an overall factor of $[i\,j]$.

Therefore, the only source of singular anti-holomorphic factors are MHV-diagrams that contain an internal propagator of momentum $q_{i+1,j} = p_{i+1} + \cdots + p_j$ which is a sum of external momenta from the collinear set such that $q^2 = s_{i+1,j} \to 0$. Hence, we conclude that only a subset of MHV-diagrams contributes to multi-collinear limits of tree amplitudes. The subset is determined by requiring that all $v-1$ internal propagators are on-shell in the multi-collinear limit. This is a powerful constraint on the types of the contributing diagrams and it simplifies the calculation.[3]

We exploit the universal nature of the splitting function by choosing to start with an amplitude with $(n+3)$ external legs, i.e. setting $N = n+3$ in eq. (3.3). The helicities of the gluons are adjusted so that the remnant 'hard' four point MHV amplitude $A_4(P^\lambda, (n+1)^+, (n+2)^{-\lambda}, (n+3)^-)$ is given by

$$A_4((n+1)^+, (n+2)^{-\lambda}, (n+3)^-, P^\lambda) = \frac{\langle n+3,\, X\rangle^4}{\langle P,\, n+1\rangle \langle n+1,\, n+2\rangle \langle n+2,\, n+3\rangle \langle n+3,\, P\rangle} \tag{3.7}$$

with $X = P$ for $\lambda = -$ and $X = n+2$ for $\lambda = +$.

To read off the collinear limits from the MHV rules, we use the limiting expressions for the spinor products: $\langle a\, q\rangle$, $\langle b\, q\rangle$ and $\langle b\, a\rangle$. Here a is a particle from the collinear set, b is a particle which is not in the collinear set, and q is the sum of the collinear momenta from $i+1$ to j. Hence, using

$$\langle a\, q\rangle = \sum_{l=i+1}^{j} \langle a\, l\rangle\, [l\, \eta]\ , \qquad \langle b\, q\rangle = \sum_{l=i+1}^{j} \langle b\, l\rangle\, [l\, \eta]\ , \tag{3.8}$$

and the expressions for spinors from the collinear set,

$$|l\rangle = \sqrt{z_l}|P\rangle\, , \qquad |l] = \sqrt{z_l}|P]\, , \qquad |a\rangle = \sqrt{z_a}|P\rangle\, , \qquad |a] = \sqrt{z_a}|P]\, , \tag{3.9}$$

we have,

$$\langle a\, q\rangle \ \to\ [P\, \eta] \sum_{l=i+1}^{j} \langle a\, l\rangle \sqrt{z_l} \equiv [P\, \eta]\, \Delta_{(1)}(i,j;a) \tag{3.10}$$

$$\langle b\, q\rangle \ \to\ [P\, \eta]\, \langle b\, P\rangle \sum_{l=i+1}^{j} z_l \tag{3.11}$$

$$\langle b\, a\rangle \ \to\ \langle b\, P\rangle \sqrt{z_a}\, . \tag{3.12}$$

Here we introduced the definition

$$\Delta_{(1)}(i,j;a) = \sum_{l=i+1}^{j} \langle a\, l\rangle \sqrt{z_l}\, . \tag{3.13}$$

Equations (3.10) and (3.11) contain a factor $[P\, \eta]$ which, however, will always cancel in expressions for relevant splitting functions. As such we can read off the collinear limits

[3] Note that this selection rule would not apply to neither gauge-fixed MHV rules (where η's are fixed to be equal to kinematic variables from the collinear set), nor to the BCF rules which mix holomorphic MHV vertices with anti-holomorphic $\overline{\text{MHV}}$ ones.

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of the amplitudes from the MHV-rules expressions by replacing terms on the left hand side of equations (3.10), (3.11) and (3.12) with the expressions on the right hand side of those equations, and further dropping the $[P\,\eta]$ factors.

Certain terms in the sums that arise in MHV rules need special attention. These are the boundary terms involving either $\langle 0\,1\rangle$ or $\langle n\,n+1\rangle$, and for these we have,

$$\frac{\langle n\,n+1\rangle}{\Delta_{(1)}(i,n;n+1)} \rightarrow -\frac{\sqrt{z_n}}{\sum_{l=i+1}^{n} z_l}\,, \tag{3.14}$$

$$\frac{\langle 0\,1\rangle}{\Delta_{(1)}(0,j;0)} \rightarrow \frac{\sqrt{z_1}}{\sum_{l=1}^{j} z_l}\,. \tag{3.15}$$

We now present our results for the splitting functions.

4. Results

In this section we give the results for the multiple collinear limit of gluons. First we give the general results for an arbitrary number of gluons with $\Delta M \leq 2$. Afterwards we give explicit results for up to four collinear gluons for all independent helicity combinations, together with some specific examples for five and six collinear gluons.

4.1 General results

In this section we present the general results for the cases where the number of gluons with negative helicity changes by at most $\Delta M = 2$, and those related by parity where the number of gluons with positive helicity changes at most by the same amount. With the help of parity these general splitting amplitudes are sufficient to obtain the explicit expressions for all helicity combinations of up to six gluons.

We will often use a more compact notation for the splitting amplitude. We denote the splitting amplitude for n collinear gluons, of which r have negative helicity, by:

$$\mathrm{split}(1^+,\ldots,m_1^-,\ldots,m_2^-,\ldots,m_r^-,\ldots,n^+ \rightarrow P^{\pm}) = \mathrm{Split}_{\pm}^{(n)}(m_1,\ldots,m_r)\,. \tag{4.1}$$

4.1.1 $\Delta M = 0$

This is the simplest case which is read directly off the single MHV vertex. The denominator of an N-point MHV amplitude is factorised as follows (in the limit of collinear $p_1,\ldots,p_n$):

$$\begin{aligned}\langle N,\,1\rangle\,\langle 1,\,2\rangle\ldots\langle n,\,n+1\rangle\ldots\langle N-1,\,N\rangle \;=\; &\left(\sqrt{z_1 z_n}\prod_{l=1}^{n-1}\langle l,\,l+1\rangle\right)\times\\ &\times\Big(\,\langle N,\,P\rangle\,\langle P,\,n+1\rangle\ldots\langle N-1,\,N\rangle\,\Big)\end{aligned} \tag{4.2}$$

where the first factor contributes to the splitting function, and the second one is the denominator of the remaining hard MHV amplitude. Hence, the splitting function is

$$\mathrm{split}(1^+,\ldots,n^+ \rightarrow P^+) = \frac{1}{\sqrt{z_1 z_n}\prod_{l=1}^{n-1}\langle l,\,l+1\rangle}\,, \tag{4.3}$$

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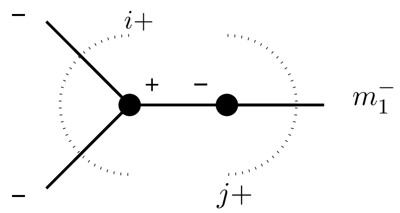

Figure 2: MHV diagrams contributing to $\mathrm{Split}_+^{(n)}(m_1)$. Negative helicity gluons are indicated by solid lines, while arbitrary numbers of positive helicity gluons emitted from each vertex are shown as dotted arcs.

and so by parity

$$\mathrm{split}(1^-, \ldots, n^- \to P^-) = \frac{(-1)^{n-1}}{\sqrt{z_1 z_n} \prod_{l=1}^{n-1} [l,\, l+1]} \,. \tag{4.4}$$

Similarly,

$$\mathrm{split}(1^+, \ldots, m_1^-, \ldots, n^+ \to P^-) = \frac{z_{m_1}^2}{\sqrt{z_1 z_n} \prod_{l=1}^{n-1} \langle l,\, l+1 \rangle} \,, \tag{4.5}$$

and

$$\mathrm{split}(1^-, \ldots, m_1^+, \ldots, n^- \to P^+) = \frac{(-1)^{n-1} z_{m_1}^2}{\sqrt{z_1 z_n} \prod_{l=1}^{n-1} [l,\, l+1]} \,. \tag{4.6}$$

4.1.2 $\Delta M = 1$

This is the next-to-MHV (NMHV) case, and in the collinear limit we need to take into account only a subset of MHV diagrams. In fact, there is only a single MHV diagram (or more precisely a single class of MHV diagrams) which can contribute to $\mathrm{Split}_+^{(n)}(m_1)$. It is shown in figure 2.[4] In the limit where gluons $1, \ldots, n$ become collinear. The left vertex in figure 2 produces a 'hard' MHV amplitude while the right vertex generates the splitting function. We need to sum over i and j in figure 2 in such a way that only diagrams with a singular propagator are selected in the collinear limit. This puts a constraint $j \leq n$ where n is the number of collinear gluons. The resulting splitting function reads,

$$\mathrm{Split}_+^{(n)}(m_1) = \frac{1}{\sqrt{z_1 z_n} \prod_{l=1}^{n-1} \langle l,\, l+1 \rangle} \left(\sum_{i=0}^{m_1-1} \sum_{j=m_1}^{n} \frac{\Delta_{(1)}(i,j;m_1)^4}{D(i,j,q_{i+1,j})} \right) , \tag{4.7}$$

where we define

$$D(i,j,q) = \frac{q_{i+1,j}^2}{\langle i,\, i+1 \rangle \langle j,\, j+1 \rangle} \Delta_{(1)}(i,j;i) \Delta_{(1)}(i,j;i+1) \Delta_{(1)}(i,j;j) \Delta_{(1)}(i,j;j+1) \,. \tag{4.8}$$

[4] MHV diagrams where hard negative helicity gluons are emitted from more than one vertex do not give rise to on-shell propagators and do not contribute in the singular limit.

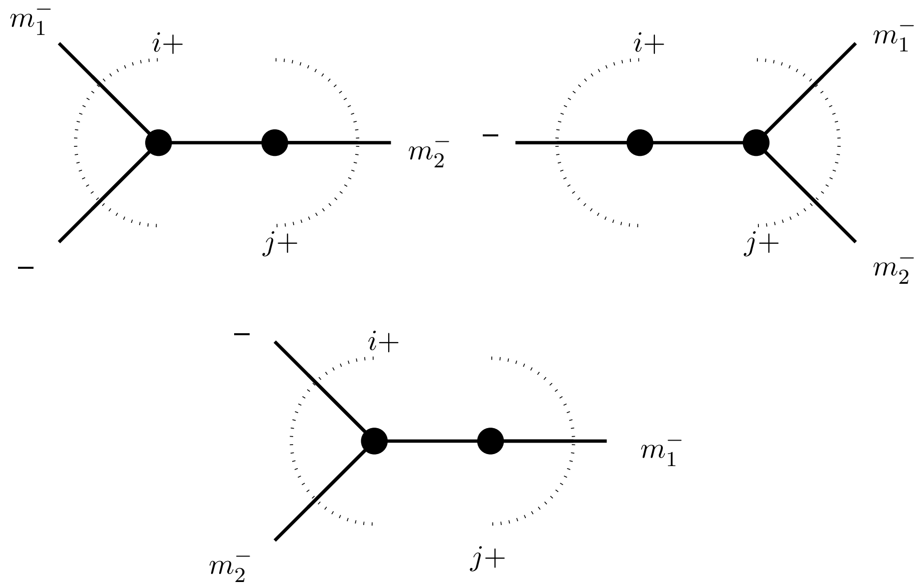

Figure 3: MHV diagrams contributing to $\mathrm{Split}_{-}^{(n)}(m_1, m_2)$.

Similarly, there are three (classes of) MHV-diagrams contributing to $\mathrm{Split}_{-}^{(n)}(m_1, m_2)$. They are shown in figure 3 and lead to a splitting function which reads

$$\begin{aligned}\mathrm{Split}_{-}^{(n)}(m_1, m_2) = & \frac{1}{\sqrt{z_1 z_n}\prod_{l=1}^{n-1}\langle l,\, l+1\rangle} \times \\ & \times \left(\sum_{i=0}^{m_1-1}\sum_{j=m_1}^{m_2-1} \frac{z_{m_2}^2 \Delta_{(1)}(i,j;m_1)^4}{D(i,j,q_{i+1,j})} + \sum_{i=m_1}^{m_2-1}\sum_{j=m_2}^{n} \frac{z_{m_1}^2 \Delta_{(1)}(i,j;m_2)^4}{D(i,j,q_{i+1,j})} + \right. \\ & \left. + \sum_{i=0}^{m_1-1}\sum_{j=m_2}^{n} \frac{\langle m_1\, m_2\rangle^4}{D(i,j,q_{i+1,j})} \left(\sum_{l=i+1}^{j} z_l \right)^4 \right). \end{aligned} \tag{4.9}$$

The remaining splitting amplitudes of the form

$$\mathrm{split}(1^-, \ldots, m_1^+, \ldots, m_2^+, \ldots, m_r^+, \ldots, n^- \to P^\pm) \tag{4.10}$$

are obtained by parity transformation through the usual replacement $\langle l,\, k\rangle \leftrightarrow -[l,\, k]$.

4.1.3 $\Delta M = 2$

The collinear limits with $\Delta M = 2$ are derived from next-to-next-to-MHV (NNMHV) diagrams. There are four (classes of) MHV-diagrams contributing to $\mathrm{Split}_{+}^{(n)}(m_1, m_2)$ which are shown in figure 4. The corresponding splitting function is,

$$\begin{aligned}\mathrm{Split}_{+}^{(n)}(m_1, m_2) = & \frac{1}{\sqrt{z_1 z_n}\prod_{l=1}^{n-1}\langle l,\, l+1\rangle} \times \\ & \times \left(\sum_{i=0}^{m_1-1}\sum_{j=m_2}^{n}\sum_{k=m_1}^{m_2-1}\sum_{r=m_2}^{j} \frac{\Delta_{(1)}(i,j;m_1)^4 \Delta_{(1)}(k,r;m_2)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})} + \right.\end{aligned}$$

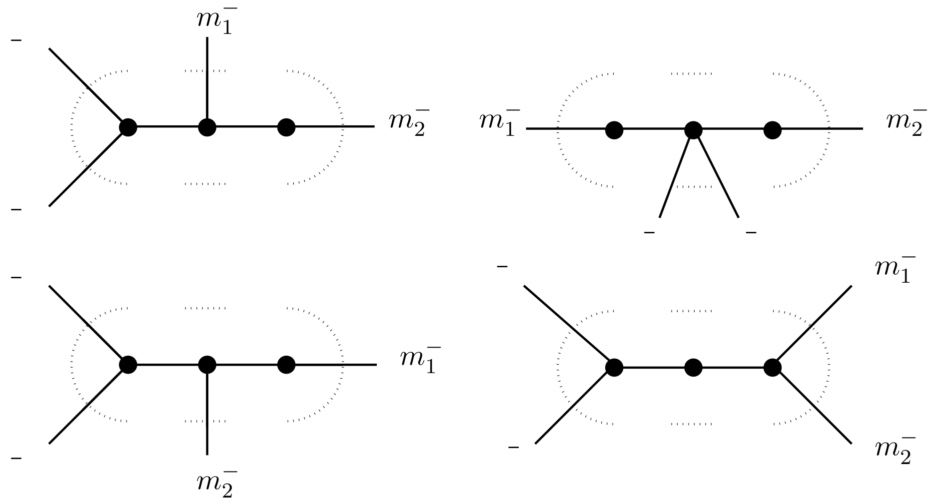

Figure 4: MHV diagrams contributing to $\mathrm{Split}_+^{(n)}(m_1, m_2)$.

$$
\begin{aligned}
&+\sum_{i=0}^{m_1-1}\sum_{j=m_1}^{k}\sum_{k=m_1}^{m_2-1}\sum_{r=m_2}^{n}\frac{\Delta_{(1)}(i,j;m_1)^4\Delta_{(1)}(k,r;m_2)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})}+\\
&+\sum_{i=0}^{k}\sum_{j=m_2}^{n}\sum_{k=0}^{m_1-1}\sum_{r=m_1}^{m_2-1}\frac{\Delta_{(1)}(i,j;m_2)^4\Delta_{(1)}(k,r;m_1)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})}+\\
&+\sum_{i=0}^{k}\sum_{j=m_2}^{n}\sum_{k=0}^{m_1-1}\sum_{r=m_2}^{j}\frac{\langle m_1\, m_2\rangle^4\,\Delta_{(2)}(i,j;k,r)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})}\Bigg)
\end{aligned}
\tag{4.11}
$$

where $\Delta_{(1)}(i,j;k)$ is given in eq. (3.13) and we introduce

$$
\Delta_{(2)}(i,j;k,r) = \sum_{u=i+1}^{j}\sum_{v=k+1}^{r}\langle u\, v\rangle\,\sqrt{z_u z_v}\,. \tag{4.12}
$$

The 'effective propagator' DD is defined by

$$
\mathrm{DD}(i,j,q_1;k,r,q_2) = \chi(i,k,q_1,q_2)\chi(r,j,q_2,q_1)\chi(j,k,q_1,q_2)D(i,j,q_1)D(k,r,q_2) \tag{4.13}
$$

in terms of D defined previously in eq. (4.8), and χ given by

$$
\chi(i,k,q_1,q_2) = \begin{cases} 1 & i \neq k \\ \frac{\Delta_{(2)}(q_1,q_2)\langle i,i+1\rangle}{\Delta_{(1)}(q_1;i+1)\Delta_{(1)}(q_2;i)} & i = k \end{cases}. \tag{4.14}
$$

Finally there are 16 classes of MHV-diagrams contributing to $\mathrm{Split}_-^{(n)}(m_1, m_2, m_3)$, coming from the 5 topologies shown in figure 5 and their cyclic permutations. The individual contributions are given by

$$
\mathrm{Split}_-^{(n)}(m_1,m_2,m_3) = \frac{1}{\sqrt{z_1 z_n}\prod_{l=1}^{n-1}\langle l,\, l+1\rangle}\sum_{i=1}^{16} A^{(i)}(m_1,m_2,m_3) \tag{4.15}
$$

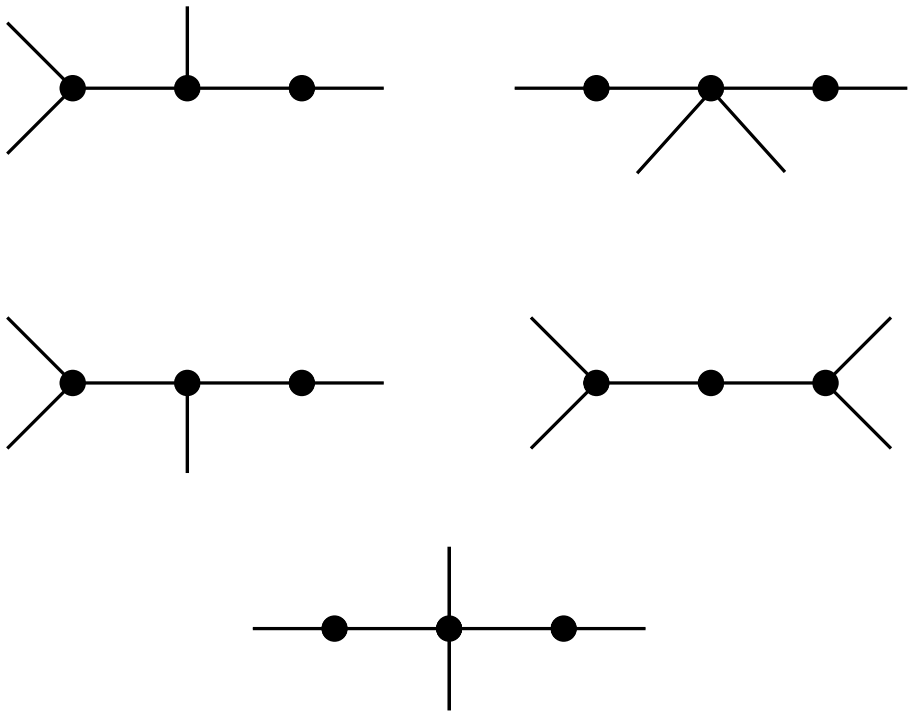

Figure 5: MHV topologies contributing to $\mathrm{Split}_{-}^{(n)}(m_1, m_2, m_3)$. The negative helicity gluons m_1, m_2 and m_3 are distributed in a cyclic way around each diagram. The remaining leg is the negative helicity gluon that remains after the collinear limit is taken.

where

$$
\begin{aligned}
A^{(1)}(m_1,m_2,m_3) &= \sum_{i=m_1}^{m_2-1}\sum_{j=m_3}^{n}\sum_{k=m_2}^{m_3-1}\sum_{r=m_3}^{j} z_{m_1}^2 \frac{\Delta_{(1)}(i,j;m_2)^4\Delta_{(1)}(k,r;m_3)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})} \\
A^{(2)}(m_1,m_2,m_3) &= \sum_{i=m_1}^{k}\sum_{j=m_3}^{n}\sum_{k=m_1}^{m_2-1}\sum_{r=m_2}^{m_3-1} z_{m_1}^2 \frac{\Delta_{(1)}(i,j;m_3)^4\Delta_{(1)}(k,r;m_2)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})} \\
A^{(3)}(m_1,m_2,m_3) &= \sum_{i=0}^{m_1-1}\sum_{j=m_2}^{m_3-1}\sum_{k=m_1}^{m_2-1}\sum_{r=m_2}^{j} z_{m_3}^2 \frac{\Delta_{(1)}(i,j;m_1)^4\Delta_{(1)}(k,r;m_2)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})} \\
A^{(4)}(m_1,m_2,m_3) &= \sum_{i=0}^{k}\sum_{j=m_2}^{m_3-1}\sum_{k=0}^{m_1-1}\sum_{r=m_1}^{m_2-1} z_{m_3}^2 \frac{\Delta_{(1)}(i,j;m_2)^4\Delta_{(1)}(k,r;m_1)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})} \\
A^{(5)}(m_1,m_2,m_3) &= \sum_{i=m_1}^{k}\sum_{j=m_3}^{n}\sum_{k=m_1}^{m_2-1}\sum_{r=m_3}^{j} z_{m_1}^2 \frac{\langle m_2\, m_3\rangle^4\,\Delta_{(2)}(i,j;k,r)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})} \\
A^{(6)}(m_1,m_2,m_3) &= \sum_{i=0}^{k}\sum_{j=m_2}^{m_3-1}\sum_{k=0}^{m_1-1}\sum_{r=m_2}^{j} z_{m_3}^2 \frac{\langle m_1\, m_2\rangle^4\,\Delta_{(2)}(i,j;k,r)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})} \\
A^{(7)}(m_1,m_2,m_3) &= \sum_{i=0}^{m_1-1}\sum_{j=m_3}^{n}\sum_{k=m_2}^{m_3-1}\sum_{r=m_3}^{j} \Big(\sum_{l=i+1}^{j} z_l\Big)^4 \frac{\langle m_1\, m_2\rangle^4\,\Delta_{(1)}(k,r;m_3)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})}
\end{aligned}
$$

$$A^{(8)}(m_1,m_2,m_3) = \sum_{i=0}^{m_1-1}\sum_{j=m_3}^{n}\sum_{k=m_1}^{m_2-1}\sum_{r=m_3}^{j}\Big(\sum_{l=i+1}^{j} z_l\Big)^4 \frac{\langle m_2\, m_3\rangle^4\,\Delta_{(1)}(k,r;m_1)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})}$$

$$A^{(9)}(m_1,m_2,m_3) = \sum_{i=0}^{k}\sum_{j=m_3}^{n}\sum_{k=0}^{m_1-1}\sum_{r=m_2}^{m_3-1}\Big(\sum_{l=i+1}^{j} z_l\Big)^4 \frac{\langle m_1\, m_2\rangle^4\,\Delta_{(1)}(k,r;m_3)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})}$$

$$A^{(10)}(m_1,m_2,m_3) = \sum_{i=0}^{m_1-1}\sum_{j=m_3}^{n}\sum_{k=m_1}^{m_2-1}\sum_{r=m_2}^{m_3-1}\Big(\sum_{l=i+1}^{j} z_l\Big)^4 \frac{\langle m_1\, m_3\rangle^4\,\Delta_{(1)}(k,r;m_2)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})}$$

$$A^{(11)}(m_1,m_2,m_3) = \sum_{i=0}^{k}\sum_{j=m_3}^{n}\sum_{k=0}^{m_1-1}\sum_{r=m_1}^{m_2-1}\Big(\sum_{l=i+1}^{j} z_l\Big)^4 \frac{\langle m_2\, m_3\rangle^4\,\Delta_{(1)}(k,r;m_1)^4}{\mathrm{DD}(i,j,q_{i+1,j};k,r,q_{k+1,r})}$$

$$A^{(12)}(m_1,m_2,m_3) = \sum_{i=m_1}^{m_2-1}\sum_{j=m_3}^{n}\sum_{k=m_2}^{r}\sum_{r=m_2}^{m_3-1} z_{m_1}^2 \frac{\Delta_{(1)}(i,k;m_2)^4\Delta_{(1)}(r,j;m_3)^4}{\mathrm{DD}(i,k,q_{i+1,k};r,j,q_{r+1,j})}$$

$$A^{(13)}(m_1,m_2,m_3) = \sum_{i=0}^{m_1-1}\sum_{j=m_2}^{m_3-1}\sum_{k=m_1}^{r}\sum_{r=m_1}^{m_2-1} z_{m_3}^2 \frac{\Delta_{(1)}(i,k;m_1)^4\Delta_{(1)}(r,j;m_2)^4}{\mathrm{DD}(i,k,q_{i+1,k};r,j,q_{r+1,j})}$$

$$A^{(14)}(m_1,m_2,m_3) = \sum_{i=0}^{m_1-1}\sum_{j=m_3}^{n}\sum_{k=m_2}^{r}\sum_{r=m_2}^{m_3-1}\Big(\sum_{l=i+1}^{k} z_l\Big)^4 \frac{\langle m_1\, m_2\rangle^4\,\Delta_{(1)}(r,j;m_3)^4}{\mathrm{DD}(i,k,q_{i+1,k};r,j,q_{r+1,j})}$$

$$A^{(15)}(m_1,m_2,m_3) = \sum_{i=0}^{m_1-1}\sum_{j=m_3}^{n}\sum_{k=m_1}^{r}\sum_{r=m_1}^{m_2-1}\Big(\sum_{l=r+1}^{j} z_l\Big)^4 \frac{\langle m_2\, m_3\rangle^4\,\Delta_{(1)}(i,k;m_1)^4}{\mathrm{DD}(i,k,q_{i+1,k};r,j,q_{r+1,j})}$$

$$A^{(16)}(m_1,m_2,m_3) = \sum_{i=0}^{m_1-1}\sum_{j=m_3}^{n}\sum_{k=m_1}^{m_2-1}\sum_{r=m_2}^{m_3-1} z_{m_2}^2 \frac{\Delta_{(1)}(i,k;m_1)^4\Delta_{(1)}(r,j;m_3)^4}{\mathrm{DD}(i,k,q_{i+1,k};r,j,q_{r+1,j})}\,.$$

4.2 Specific results for $n < 7$

In this section we present compact expressions for splitting amplitudes with up to six collinear gluons. These results are obtained directly from the general expressions given in section 4.1.

First we note that splitting amplitudes satisfy reflection symmetry,

$$\mathrm{split}(1^{\lambda_1},\ldots,n^{\lambda_n}\to P^{\pm}) = (-1)^{n+1}\mathrm{split}(n^{\lambda_n},\ldots,1^{\lambda_1}\to P^{\pm}) \tag{4.16}$$

and the dual Ward identity, see e.g. [64],

$$\begin{aligned}&\mathrm{split}(1^{\lambda_1},2^{\lambda_2},\ldots,n^{\lambda_n}\to P^{\pm}) + \mathrm{split}(2^{\lambda_2},1^{\lambda_1},\ldots,n^{\lambda_n}\to P^{\pm}) + \cdots\\ &\quad + \mathrm{split}(2^{\lambda_2},\ldots,1^{\lambda_1},n^{\lambda_n}\to P^{\pm}) + \mathrm{split}(2^{\lambda_2},\ldots,n^{\lambda_n},1^{\lambda_1}\to P^{\pm}) = 0\,.\end{aligned} \tag{4.17}$$

These relations reduce the number of independent splitting amplitudes significantly.

4.2.1 $n = 2$

For two collinear gluons there are two independent splitting amplitudes with $\Delta M = 0$. All others can be obtained by parity and reflection. Setting $z_1 = z$ and $z_2 = (1-z)$, we find

$$\mathrm{split}(1^+,2^+\to P^+) = \frac{1}{\sqrt{z(1-z)}\,\langle 12\rangle}\,, \tag{4.18}$$

$$\mathrm{split}(1^-, 2^+ \to P^-) = \frac{z^2}{\sqrt{z(1-z)}\,\langle 12\rangle}\,. \tag{4.19}$$

As expected, the splitting amplitudes have a single pole proportional to $\langle 1\,2\rangle^{-1}$. Note that in the soft limit $z \to 0$, we see that helicity conservation ensures that $\mathrm{split}(1^-, 2^+ \to P^-) \to 0$.

4.2.2 $n = 3$ result from MHV rules

For three collinear gluons there are three independent splitting amplitudes with $\Delta M = 0$. They all follow directly from a single MHV vertex and are given by

$$\begin{aligned}
\mathrm{split}(1^+, 2^+, 3^+ \to P^+) &= \frac{1}{\sqrt{z_1 z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle}\,,\\
\mathrm{split}(1^-, 2^+, 3^+ \to P^-) &= \frac{z_1^2}{\sqrt{z_1 z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle}\,,\\
\mathrm{split}(1^+, 2^-, 3^+ \to P^-) &= \frac{z_2^2}{\sqrt{z_1 z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle}\,.
\end{aligned}$$

Parity and the reflection symmetry, $\mathrm{split}(1^+, 2^+, 3^- \to P^-) = \mathrm{split}(3^-, 2^+, 1^+ \to P^-)$, give the rest.

When $\Delta M = 1$, there are three amplitudes,

$$\begin{aligned}
\mathrm{split}(1^-, 2^+, 3^+ \to P^+) &= \frac{\langle 1\,2\rangle\, {z_2}^2}{\sqrt{z_1 z_2 z_3}\, s_{1,2}\,(z_1+z_2)\left(\langle 1\,3\rangle\sqrt{z_1} + \langle 2\,3\rangle\sqrt{z_2}\right)}\\
&\quad + \frac{\left(\langle 1\,2\rangle\sqrt{z_2} + \langle 1\,3\rangle\sqrt{z_3}\right)^3}{s_{1,3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\left(\langle 1\,3\rangle\sqrt{z_1} + \langle 2\,3\rangle\sqrt{z_2}\right)}\,, \qquad (4.20)\\
\mathrm{split}(1^+, 2^-, 3^+ \to P^+) &= -\mathrm{split}(2^-, 1^+, 3^+ \to P^+) - \mathrm{split}(1^+, 3^+, 2^- \to P^+)\\
&= \frac{\langle 1\,2\rangle\, {z_1}^2}{\sqrt{z_1 z_2 z_3}\, s_{1,2}\,(z_1+z_2)\left(\langle 1\,3\rangle\sqrt{z_1} + \langle 2\,3\rangle\sqrt{z_2}\right)} +\\
&\quad + \frac{\left(\langle 2\,1\rangle\sqrt{z_1} + \langle 2\,3\rangle\sqrt{z_3}\right)^4}{s_{1,3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\left(\langle 1\,3\rangle\sqrt{z_1} + \langle 2\,3\rangle\sqrt{z_2}\right)\left(\langle 1\,2\rangle\sqrt{z_2} + \langle 1\,3\rangle\sqrt{z_3}\right)} +\\
&\quad + \frac{\langle 2\,3\rangle\, {z_3}^2}{\sqrt{z_1 z_2 z_3}\, s_{2,3}\,(z_2+z_3)\left(\langle 1\,2\rangle\sqrt{z_2} + \langle 1\,3\rangle\sqrt{z_3}\right)}\,, \qquad (4.21)\\
\mathrm{split}(1^+, 2^+, 3^- \to P^+) &= \mathrm{split}(3^-, 2^+, 1^+ \to P^+)\,. \qquad (4.22)
\end{aligned}$$

In addition to singular terms like $\langle 1\,2\rangle$, we see that the splitting functions contain mixed terms like $s_{1,3}$. The net singularity is schematically of the form $[\ ]\langle\ \rangle$.

Note that $\mathrm{split}(1^-, 2^+, 3^+ \to P^+)$ contains poles in $s_{1,2}$ and the triple invariant $s_{1,3} = s_{123}$ but not in $s_{2,3}$. This is because there is no MHV rule graph with a three-point vertex involving two positive helicity gluons.

Expressions for these splitting functions are given in eq. (5.52) of ref. [64]. The results given here are more compact and have a rather different analytic form. After adjusting the normalisation of the colour matrices, the splitting functions of eqs. (4.20)–(4.21) numerically agree with those of ref. [64].

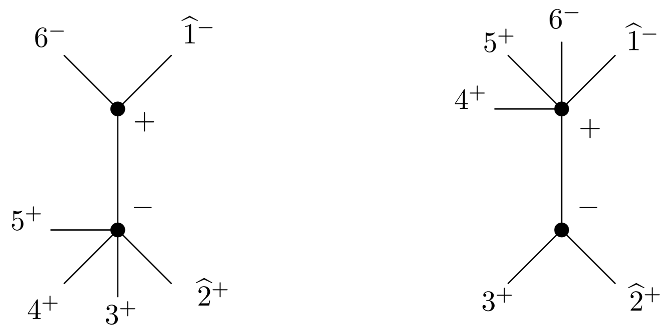

Figure 6: BCF diagrams contributing to $A(1^-, 2^+, 3^+, 4^+, 5^-, 6^-)$.

4.2.3 $n = 3$ result from the BCF recursion relation

We now want to rederive the above results using the BCF recursion relation of [3]. In doing this we will (a) draw some useful comparisons between the 'BCF recursion' and the 'MHV rules' formalisms from the perspective of collinear amplitudes; and (b) test our expressions, such as eq. (4.20) for split$(1^-, 2^+, 3^+ \to P^+)$.

We start with the six-point amplitude $A(1^-, 2^+, 3^+, 4^+, 5^-, 6^-)$, and calculate it via the BCF recursive approach. We ultimately want to take the collinear limit $1 \,||\, 2 \,||\, 3 \to P^+$, so it will be convenient to choose the 'marked' gluons (required for the BCF recursive set-up) to be from this collinear set. Hence, we will mark the $\hat{1}^-$ and $\hat{2}^+$ gluons. There are only two BCF diagrams which contribute to the full amplitude, and they are shown in figure 6. We now note that in this particular collinear limit, only the second of these diagrams contains an on-shell propagator, $1/s_{23}$. Nevertheless, in distinction with the MHV rules approach which we have adopted previously, both BCF diagrams need to be taken into account in the collinear limit.

The full amplitude reads

$$A(1^-, 2^+, 3^+, 4^+, 5^-, 6^-) = \frac{1}{\langle 3|1+2|6]} \left(\frac{\langle 5|6+1|2]^3}{[6\,1][1\,2]\langle 3\,4\rangle\langle 4\,5\rangle s_{3,5}} + \frac{\langle 1|2+3|4]^3}{[4\,5][5\,6]\langle 1\,2\rangle\langle 2\,3\rangle s_{1,3}} \right), \tag{4.23}$$

where the two terms on the right hand side correspond to the two BCF diagrams above (cf. eq. (2.9) of ref. [3]).

In the $1 \,||\, 2 \,||\, 3 \to P^+$ collinear limit, the first term becomes

$$\frac{\langle 1\,2\rangle z_2^2}{\sqrt{z_1 z_2 z_3}\, s_{1,2}(z_1+z_2)(\langle 1\,3\rangle\sqrt{z_1} + \langle 2\,3\rangle\sqrt{z_2})} \times \frac{\langle 5\,6\rangle^4}{\langle 5\,6\rangle\langle 6\,P\rangle\langle P\,4\rangle\langle 4\,5\rangle}\,. \tag{4.24}$$

This term factors into a contribution to the splitting amplitudes multiplied by a four-point MHV vertex. In contrast, in the collinear limit the second term factors onto the $\overline{\text{MHV}}$ type diagram, written in terms of the anti-holomorphic spinor products,

$$\frac{(\langle 1\,2\rangle\sqrt{z_2} + \langle 1\,3\rangle\sqrt{z_3})^3}{s_{1,3}\langle 1\,2\rangle\langle 2\,3\rangle(\langle 1\,3\rangle\sqrt{z_1} + \langle 2\,3\rangle\sqrt{z_2})} \times \frac{[P\,4]^4}{[P\,4][4\,5][5\,6][6\,P]}\,. \tag{4.25}$$

For the special case of four-point amplitudes, the $\overline{\text{MHV}}$ and MHV amplitudes coincide and we find an identical result to eq. (4.20).

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Likewise, to test our expression for $\text{split}(1^+, 2^-, 3^+ \to P^+)$ we start from eq. (3.4) in [3];

$$A(1^+, 2^-, 3^+, 4^-, 5^+, 6^-) = \frac{[1\,3]^4 \langle 4\,6\rangle^4}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle s_{1,3}\langle 6|1+2|3]\langle 4|2+3|1]} + \\ + \frac{\langle 2\,6\rangle^4 [3\,5]^4}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]s_{3,5}\langle 6|4+5|3]\langle 2|3+4|5]} + \\ + \frac{[1\,5]^4 \langle 2\,4\rangle^4}{\langle 2\,3\rangle\langle 3\,4\rangle[5\,6][6\,1]s_{2,4}\langle 4|2+3|1]\langle 2|3+4|5]} . \tag{4.26}$$

Taking the collinear limit $1\,||\,2\,||\,3 \to P^+$, we find that

$$\text{Split}(1^+, 2^-, 3^+ \to P^+) = \frac{z_2^2 z_3^2 [1\,2]}{\sqrt{z_1 z_2 z_3} s_{1,2}(z_1+z_2)\left([1\,3]\sqrt{z_1} + [2\,3]\sqrt{z_2}\right)} + \\ + \frac{[1\,3]^4}{s_{1,3}[1\,2][2\,3]\left([1\,3]\sqrt{z_1} + [2\,3]\sqrt{z_2}\right)\left([1\,2]\sqrt{z_2} + [1\,3]\sqrt{z_3}\right)} + \\ + \frac{z_1^2 z_2^2 [2\,3]}{\sqrt{z_1 z_2 z_3} s_{2,3}(z_2+z_3)\left([1\,2]\sqrt{z_2} + [1\,3]\sqrt{z_3}\right)} . \tag{4.27}$$

This result has the same kinematic-invariant pole structure as eq. (4.21), but otherwise is not obviously equivalent to eq. (4.21). Note that eq. (4.27) contains terms like $\left([1\,2]\sqrt{z_2} + [1\,3]\sqrt{z_3}\right)$ (rather than $\left(\langle 1\,2\rangle\sqrt{z_2} + \langle 1\,3\rangle\sqrt{z_3}\right)$). Despite appearances, a more careful (e.g. numerical) comparison shows that these two results, eqs. (4.21) and (4.27), are in fact the same.

4.2.4 $n = 4$

For $n = 4$, there are five collinear limits coming directly from MHV amplitudes where the number of gluons with negative helicity doesn't change, $\Delta M = 0$,

$$\text{split}(1^+, 2^+, 3^+, 4^+ \to P^+) = \frac{1}{\sqrt{z_1 z_4}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle} , \tag{4.28}$$

$$\text{split}(1^-, 2^+, 3^+, 4^+ \to P^-) = \frac{z_1^2}{\sqrt{z_1 z_4}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle} , \tag{4.29}$$

$$\text{split}(1^+, 2^-, 3^+, 4^+ \to P^-) = \frac{z_2^2}{\sqrt{z_1 z_4}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle} . \tag{4.30}$$

The remaining two are obtained by reflection symmetry,

$$\text{split}(1^+, 2^+, 3^-, 4^+ \to P^-) = -\text{split}(4^+, 3^-, 2^+, 1^+ \to P^-) , \tag{4.31}$$

$$\text{split}(1^+, 2^+, 3^+, 4^- \to P^-) = -\text{split}(4^-, 3^+, 2^+, 1^+ \to P^-) . \tag{4.32}$$

When $\Delta M = 1$, there are ten splitting amplitudes however only three are imdependent,

$$\text{split}(1^-, 2^+, 3^+, 4^+ \to P^+) = \mathcal{B}_1(1,2,3,4) \\ = -\frac{z_2^{3/2}\,\langle 1\,2\rangle}{\sqrt{z_1 z_4}\,\langle 3\,4\rangle\, s_{1,2}\,(z_1+z_2)\,\Delta_{(1)}(0,2;3)} +$$

$$+\frac{\Delta_{(1)}(0,3;1)^3}{\sqrt{z_4}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\, s_{1,3}(z_1{+}z_2{+}z_3)\Delta_{(1)}(0,3;3)\Delta_{(1)}(0,3;4)}-$$
$$-\frac{\Delta_{(1)}(0,4;1)^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\, s_{1,4}\Delta_{(1)}(0,4;4)}, \tag{4.33}$$

$$\begin{aligned}\mathrm{split}(1^-,2^-,3^+,4^+\to P^-) &= \mathcal{B}_2(1,2,3,4)\\ &= -\frac{z_1^{3/2} z_3^{3/2}\,\langle 2\,3\rangle}{\sqrt{z_2 z_4} s_{2,3}\Delta_{(1)}(1,3;1)\Delta_{(1)}(1,3;4)}-\\ &\quad -\frac{{z_1}^{3/2}\Delta_{(1)}(1,4;2)^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\, s_{2,4}\Delta_{(1)}(1,4;1)\Delta_{(1)}(1,4;4)\,(1-z_1)}-\\ &\quad -\frac{\langle 1\,2\rangle\,(z_1+z_2)^3}{\sqrt{z_1 z_2 z_4}\,\langle 3\,4\rangle\, s_{1,2}\Delta_{(1)}(0,2;3)}+\\ &\quad +\frac{\langle 1\,2\rangle^3\,(1-z_4)^3}{\sqrt{z_4}\,\langle 2\,3\rangle\, s_{1,3}\Delta_{(1)}(0,3;1)\Delta_{(1)}(0,3;3)\Delta_{(1)}(0,3;4)}-\\ &\quad -\frac{\langle 1\,2\rangle^3}{s_{1,4}\Delta_{(1)}(0,4;1)\Delta_{(1)}(0,4;4)\,\langle 2\,3\rangle\,\langle 3\,4\rangle}\,,\end{aligned} \tag{4.34}$$

and,

$$\begin{aligned}\mathrm{split}(1^-,2^+,3^-,4^+\to P^-) &= \mathcal{B}_3(1,2,3,4)\\ &= -\frac{z_2^{3/2}{z_3}^2\,\langle 1\,2\rangle}{\sqrt{z_1 z_4}\,\langle 3\,4\rangle\, s_{1,2}\,(z_1+z_2)\,\Delta_{(1)}(0,2;3)}-\\ &\quad -\frac{z_1^{3/2} z_2^{3/2}\,\langle 2\,3\rangle}{\sqrt{z_3 z_4} s_{2,3}\Delta_{(1)}(1,3;1)\Delta_{(1)}(1,3;4)}-\\ &\quad -\frac{{z_1}^{3/2}\Delta_{(1)}(1,4;3)^4}{\langle 2\,3\rangle\,\langle 3\,4\rangle\, s_{2,4}\Delta_{(1)}(1,4;1)\Delta_{(1)}(1,4;2)\Delta_{(1)}(1,4;4)(1-z_1)}-\\ &\quad -\frac{{z_1}^{3/2}\Delta_{(1)}(2,4;3)^3}{\langle 1\,2\rangle\,\langle 3\,4\rangle\, s_{3,4}\Delta_{(1)}(2,4;2)\Delta_{(1)}(2,4;4)\,(z_3+z_4)}+\\ &\quad +\frac{\langle 1\,3\rangle^4\,(z_1+z_2+z_3)^3}{\sqrt{z_4}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\, s_{1,3}\Delta_{(1)}(0,3;1)\Delta_{(1)}(0,3;3)\Delta_{(1)}(0,3;4)}-\\ &\quad -\frac{\langle 1\,3\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\, s_{1,4}\Delta_{(1)}(0,4;4)\Delta_{(1)}(0,4;1)}\,,\end{aligned} \tag{4.35}$$

where $\Delta_{(1)}(i,j;k)$ is given in eq. (3.13). The seven remaining $\Delta M = 1$ splitting functions can be obtained by using the dual ward identity,

$$\begin{aligned}\mathrm{split}(1^+,2^-,3^+,4^+\to P^+) &= -\mathcal{B}_1(2,1,3,4)-\mathcal{B}_1(2,3,1,4)-\mathcal{B}_1(2,3,4,1)\,,\\ \mathrm{split}(1^+,2^+,3^-,4^+\to P^+) &= \mathcal{B}_1(3,4,2,1)+\mathcal{B}_1(3,2,4,1)+\mathcal{B}_1(3,2,1,4)\,,\\ \mathrm{split}(1^-,2^+,3^+,4^-\to P^-) &= \mathcal{B}_3(4,3,1,2)+\mathcal{B}_2(4,1,3,2)+\mathcal{B}_2(1,4,3,2)\,,\\ \mathrm{split}(1^+,2^-,3^-,4^+\to P^-) &= -\mathcal{B}_3(2,1,3,4)-\mathcal{B}_2(2,3,1,4)-\mathcal{B}_2(2,3,4,1)\,,\end{aligned} \tag{4.36}$$

or reflection symmetry,

$$\begin{aligned}
\mathrm{split}(1^+,2^+,3^+,4^- \to P^+) &= -\mathrm{split}(4^-,3^+,2^+,1^+ \to P^-)\,,\\
\mathrm{split}(1^+,2^-,3^+,4^- \to P^-) &= -\mathrm{split}(4^-,3^+,2^-,1^+ \to P^-)\,,\\
\mathrm{split}(1^+,2^+,3^-,4^- \to P^-) &= -\mathrm{split}(4^-,3^-,2^+,1^+ \to P^-)\,.
\end{aligned} \tag{4.37}$$

Finally splitting functions with $\Delta M = 2,\ 3$ are related to those given above by the parity transformation.

Inspection of eqs. (4.33), (4.34) and (4.35) reveals that each term is inversely proportional to a single invariant, in keeping with its MHV rules origins. For this type of collinear limit, there are potentially six invariants, the double invariants $s_{1,2}, s_{2,3}, s_{3,4}$, the triple invariants $s_{1,3}, s_{2,4}$ and $s_{1,4}$. Some poles are absent because the MHV rules forbid that type of contribution. For example, in $\mathrm{split}(1^-,2^+,3^+,4^+ \to P^+)$, there are no contributions with poles in $s_{2,3}$, $s_{3,4}$ or $s_{2,4}$ precisely because these poles correspond to forbidden MHV diagrams.

Expressions for the four gluon splitting functions are given in ref. [64]. The results given here are more compact and have a rather different analytic form. After adjusting the normalisation of the colour matrices, the splitting functions of eqs. (4.33)–(4.35) numerically agree with those of ref. [64].

4.2.5 $n = 5$

In total there are 64 different splitting amplitudes, but only eleven are independent. The rest can be obtained with the help of parity, reflection and dual ward identities. The three simplest independent collinear limits can be obtained using only MHV rules,

$$\begin{aligned}
\mathrm{split}(1^+,2^+,3^+,4^+,5^+ \to P^+) &= \frac{1}{\sqrt{z_1 z_5}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle}\,,\\
\mathrm{split}(1^-,2^+,3^+,4^+,5^+ \to P^-) &= \frac{z_1^2}{\sqrt{z_1 z_5}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle}\,,\\
\mathrm{split}(1^+,2^-,3^+,4^+,5^+ \to P^-) &= \frac{z_2^2}{\sqrt{z_1 z_5}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle}\,.
\end{aligned}$$

The amplitudes with $\Delta M = 1$ require the application of eqs. (4.7) and (4.9). There are 5 independent amplitudes in this class of splitting amplitudes, but we give here only two examples, one for each of the cases $- \to +$ and $-- \to -$,

$$\begin{aligned}
&\mathrm{split}(1^-,2^+,3^+,4^+,5^+ \to P^+) =\\
&= \frac{\left(\Delta_{(1)}(0,2;1)\right)^3 \sqrt{z_1}}{\sqrt{z_1 z_5}\,\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\, s_{1,2}\,(z_1+z_2)\,\Delta_{(1)}(0,2;2)\Delta_{(1)}(0,2;3)} +\\
&\quad + \frac{\left(\Delta_{(1)}(0,3;1)\right)^3 \sqrt{z_1}}{\sqrt{z_1 z_5}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle\, s_{1,3}\,(z_1+z_2+z_3)\,\Delta_{(1)}(0,3;3)\Delta_{(1)}(0,3;4)} +\\
&\quad + \frac{\left(\Delta_{(1)}(0,4;1)\right)^3 \sqrt{z_1}}{\sqrt{z_1 z_5}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\, s_{1,4}\,(z_1+z_2+z_3+z_4)\,\Delta_{(1)}(0,4;4)\Delta_{(1)}(0,4;5)} -
\end{aligned}$$

$$- \frac{\left(\Delta_{(1)}(0,5;1)\right)^3}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle\, s_{1,5}\Delta_{(1)}(0,5;5)}\,, \tag{4.38}$$

and,

$$\begin{aligned}
&\text{split}(1^-,2^-,3^+,4^+,5^+ \to P^-) = \\
&= \frac{{z_1}^2 \left(\Delta_{(1)}(1,3;2)\right)^3}{\sqrt{z_1 z_5}\, \langle 2\,3\rangle \langle 4\,5\rangle\, s_{2,3}\Delta_{(1)}(1,3;1)\Delta_{(1)}(1,3;3)\Delta_{(1)}(1,3;4)} + \\
&+ \frac{{z_1}^2 \left(\Delta_{(1)}(1,4;2)\right)^3}{\sqrt{z_1 z_5}\, \langle 2\,3\rangle \langle 3\,4\rangle\, s_{2,4}\Delta_{(1)}(1,4;1)\Delta_{(1)}(1,4;4)\Delta_{(1)}(1,4;5)} - \\
&- \frac{{z_1}^2 \left(\Delta_{(1)}(1,5;2)\right)^3 \sqrt{z_5}}{\sqrt{z_1 z_5}\, \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle\, s_{2,5}\Delta_{(1)}(1,5;1)\Delta_{(1)}(1,5;5)\,(z_2+z_3+z_4+z_5)} + \\
&+ \frac{(\langle 1\,2\rangle)^3 (z_1+z_2)^3 \sqrt{z_1}}{\sqrt{z_1 z_5}\, \langle 3\,4\rangle \langle 4\,5\rangle\, s_{1,2}\Delta_{(1)}(0,2;1)\Delta_{(1)}(0,2;2)\Delta_{(1)}(0,2;3)} + \\
&+ \frac{(\langle 1\,2\rangle)^3 (z_1+z_2+z_3)^3 \sqrt{z_1}}{\sqrt{z_1 z_5}\, \langle 2\,3\rangle \langle 4\,5\rangle\, s_{1,3}\Delta_{(1)}(0,3;1)\Delta_{(1)}(0,3;3)\Delta_{(1)}(0,3;4)} + \\
&+ \frac{(\langle 1\,2\rangle)^3 (z_1+z_2+z_3+z_4)^3 \sqrt{z_1}}{\sqrt{z_1 z_5}\, \langle 2\,3\rangle \langle 3\,4\rangle\, s_{1,4}\Delta_{(1)}(0,4;1)\Delta_{(1)}(0,4;4)\Delta_{(1)}(0,4;5)} - \\
&- \frac{(\langle 1\,2\rangle)^3}{\langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle\, s_{1,5}\Delta_{(1)}(0,5;1)\Delta_{(1)}(0,5;5)}\,.
\end{aligned} \tag{4.39}$$

The most complicated amplitudes are those with $\Delta M = 2$ and require the use of eq. (4.11). There are three independent splitting functions, but here we only give one example.

$$\begin{aligned}
&\text{split}(1^-,2^-,3^+,4^+,5^+ \to P^+) = \\
&= - \frac{\left(\Delta_{(1)}(0,3;1)\right)^3 \left(\Delta_{(1)}(1,3;2)\right)^3}{\sqrt{z_5}\, \langle 2\,3\rangle \langle 4\,5\rangle\, \Delta_{(2)}(0,3;1,3) s_{1,3}\,(z_1+z_2+z_3)\,\Delta_{(1)}(0,3;4) s_{2,3}\Delta_{(1)}(1,3;1)\Delta_{(1)}(1,3;3)} + \\
&+ \frac{\left(\Delta_{(1)}(0,4;1)\right)^3 \left(\Delta_{(1)}(1,3;2)\right)^3}{\sqrt{z_5}\, \langle 2\,3\rangle\, s_{1,4}\,(1-z_5)\,\Delta_{(1)}(0,4;4)\Delta_{(1)}(0,4;5) s_{2,3}\Delta_{(1)}(1,3;1)\Delta_{(1)}(1,3;3)\Delta_{(1)}(1,3;4)} - \\
&- \frac{\left(\Delta_{(1)}(0,4;1)\right)^3 \left(\Delta_{(1)}(1,4;2)\right)^3}{\sqrt{z_5}\, \langle 2\,3\rangle \langle 3\,4\rangle\, \Delta_{(2)}(0,4;1,4) s_{1,4}\,(1-z_5)\,\Delta_{(1)}(0,4;5) s_{2,4}\Delta_{(1)}(1,4;1)\Delta_{(1)}(1,4;4)} - \\
&- \frac{\left(\Delta_{(1)}(0,5;1)\right)^3 \left(\Delta_{(1)}(1,3;2)\right)^3}{\Delta_{(1)}(1,3;4)\, \langle 2\,3\rangle \langle 4\,5\rangle\, s_{1,5}\Delta_{(1)}(0,5;5)\Delta_{(1)}(1,3;1)\Delta_{(1)}(1,3;3) s_{2,3}} - \\
&- \frac{\left(\Delta_{(1)}(1,4;2)\right)^3 \left(\Delta_{(1)}(0,5;1)\right)^3}{s_{2,4}\Delta_{(1)}(1,4;1)\Delta_{(1)}(1,4;4)\Delta_{(1)}(1,4;5)\, \langle 2\,3\rangle \langle 3\,4\rangle\, s_{1,5}\Delta_{(1)}(0,5;5)} + \\
&+ \frac{\left(\Delta_{(1)}(1,5;2)\right)^3 \left(\Delta_{(1)}(0,5;1)\right)^3}{\Delta_{(2)}(0,5;1,5)\Delta_{(1)}(1,5;1)\Delta_{(1)}(1,5;5) s_{2,5}\, \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle\, s_{1,5}} + \\
&+ \frac{(\langle 1\,2\rangle)^3 \left(\Delta_{(2)}(0,3;0,2)\right)^3}{\sqrt{z_5}\, \langle 4\,5\rangle\, s_{1,3}(z_1+z_2+z_3)\Delta_{(1)}(0,3;3)\Delta_{(1)}(0,3;4) s_{1,2}\Delta_{(1)}(0,2;1)\Delta_{(1)}(0,2;2)\Delta_{(1)}(0,2;3)} + \\
&+ \frac{(\langle 1\,2\rangle)^3 \left(\Delta_{(2)}(0,4;0,2)\right)^3}{\sqrt{z_5}\, \langle 3\,4\rangle\, s_{1,4}\,(1-z_5)\,\Delta_{(1)}(0,4;4)\Delta_{(1)}(0,4;5) s_{1,2}\Delta_{(1)}(0,2;1)\Delta_{(1)}(0,2;2)\Delta_{(1)}(0,2;3)} +
\end{aligned}$$

$$
+\frac{(\langle 1\,2\rangle)^3\left(\Delta_{(2)}(0,4;0,3)\right)^3}{\sqrt{z_5}\,\langle 2\,3\rangle\, s_{1,4}\,(1-z_5)\,\Delta_{(1)}(0,4;4)\Delta_{(1)}(0,4;5)s_{1,3}\Delta_{(1)}(0,3;1)\Delta_{(1)}(0,3;3)\Delta_{(1)}(0,3;4)}-
$$
$$
-\frac{(\langle 1\,2\rangle)^3\left(\Delta_{(2)}(0,5;0,2)\right)^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\, s_{1,5}\Delta_{(1)}(0,5;5)s_{1,2}\Delta_{(1)}(0,2;1)\Delta_{(1)}(0,2;2)\Delta_{(1)}(0,2;3)}-
$$
$$
-\frac{\left(\Delta_{(2)}(0,5;0,3)\right)^3(\langle 1\,2\rangle)^3}{\Delta_{(1)}(0,3;4)s_{1,3}\Delta_{(1)}(0,3;1)\,\langle 2\,3\rangle\,\langle 4\,5\rangle\, s_{1,5}\Delta_{(1)}(0,5;5)\Delta_{(1)}(0,3;3)}-
$$
$$
-\frac{\left(\Delta_{(2)}(0,5;0,4)\right)^3(\langle 1\,2\rangle)^3}{\Delta_{(1)}(0,4;5)\Delta_{(1)}(0,4;1)\,\langle 2\,3\rangle\,\langle 3\,4\rangle\, s_{1,5}\Delta_{(1)}(0,5;5)s_{1,4}\Delta_{(1)}(0,4;4)}. \tag{4.40}
$$

4.2.6 $n = 6$

Finally, for six collinear gluons there are $2^7 = 128$ different splitting amplitudes, which can be expressed by 23 independent ones. To find all independent amplitudes we have to use eq. (4.15) for the first time. Due to the length of the results we give here only two examples obtained with the help of eqs. (4.7) and (4.9),

$$
\mathrm{split}(1^-,2^+,3^+,4^+,5^+,6^+\to P^+) =
$$
$$
=\frac{\left(\Delta_{(1)}(0,2;1)\right)^3\sqrt{z_1}}{\sqrt{z_1 z_6}\,\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\, s_{1,2}\,(z_1+z_2)\,\Delta_{(1)}(0,2;2)\Delta_{(1)}(0,2;3)}+
$$
$$
+\frac{\left(\Delta_{(1)}(0,3;1)\right)^3\sqrt{z_1}}{\sqrt{z_1 z_6}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\, s_{1,3}\,(z_1+z_2+z_3)\,\Delta_{(1)}(0,3;3)\Delta_{(1)}(0,3;4)}+
$$
$$
+\frac{\left(\Delta_{(1)}(0,4;1)\right)^3\sqrt{z_1}}{\sqrt{z_1 z_6}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 5\,6\rangle\, s_{1,4}\,(z_1+z_2+z_3+z_4)\,\Delta_{(1)}(0,4;4)\Delta_{(1)}(0,4;5)}+
$$
$$
+\frac{\left(\Delta_{(1)}(0,5;1)\right)^3\sqrt{z_1}}{\sqrt{z_1 z_6}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\, s_{1,5}\,(z_1+z_2+z_3+z_4+z_5)\,\Delta_{(1)}(0,5;5)\Delta_{(1)}(0,5;6)}-
$$
$$
-\frac{\left(\Delta_{(1)}(0,6;1)\right)^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\, s_{1,6}\Delta_{(1)}(0,6;6)}\,, \tag{4.41}
$$
$$
\mathrm{split}(1^-,2^-,3^+,4^+,5^+,6^+\to P^-) =
$$
$$
=\frac{{z_1}^2\left(\Delta_{(1)}(1,3;2)\right)^3}{\sqrt{z_1 z_6}\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\, s_{2,3}\Delta_{(1)}(1,3;1)\Delta_{(1)}(1,3;3)\Delta_{(1)}(1,3;4)}+
$$
$$
+\frac{{z_1}^2\left(\Delta_{(1)}(1,4;2)\right)^3}{\sqrt{z_1 z_6}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 5\,6\rangle\, s_{2,4}\Delta_{(1)}(1,4;1)\Delta_{(1)}(1,4;4)\Delta_{(1)}(1,4;5)}+
$$
$$
+\frac{{z_1}^2\left(\Delta_{(1)}(1,5;2)\right)^3}{\sqrt{z_1 z_6}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\, s_{2,5}\Delta_{(1)}(1,5;1)\Delta_{(1)}(1,5;5)\Delta_{(1)}(1,5;6)}-
$$
$$
-\frac{{z_1}^2\left(\Delta_{(1)}(1,6;2)\right)^3\sqrt{z_6}}{\sqrt{z_1 z_6}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\, s_{2,6}\Delta_{(1)}(1,6;1)\Delta_{(1)}(1,6;6)\,(z_2+z_3+z_4+z_5+z_6)}+
$$
$$
+\frac{(\langle 1\,2\rangle)^3\,(z_1+z_2)^3\sqrt{z_1}}{\sqrt{z_1 z_6}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\, s_{1,2}\Delta_{(1)}(0,2;1)\Delta_{(1)}(0,2;2)\Delta_{(1)}(0,2;3)}+
$$

$$
\begin{aligned}
&+\frac{(\langle 1\,2\rangle)^3\,(z_1+z_2+z_3)^3\,\sqrt{z_1}}{\sqrt{z_1 z_6}\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,s_{1,3}\Delta_{(1)}(0,3;1)\Delta_{(1)}(0,3;3)\Delta_{(1)}(0,3;4)}+\\
&+\frac{(\langle 1\,2\rangle)^3\,(z_1+z_2+z_3+z_4)^3\,\sqrt{z_1}}{\sqrt{z_1 z_6}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 5\,6\rangle\,s_{1,4}\Delta_{(1)}(0,4;1)\Delta_{(1)}(0,4;4)\Delta_{(1)}(0,4;5)}+\\
&+\frac{(\langle 1\,2\rangle)^3\,(z_1+z_2+z_3+z_4+z_5)^3\,\sqrt{z_1}}{\sqrt{z_1 z_6}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,s_{1,5}\Delta_{(1)}(0,5;1)\Delta_{(1)}(0,5;5)\Delta_{(1)}(0,5;6)}-\\
&-\frac{(\langle 1\,2\rangle)^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,s_{1,6}\Delta_{(1)}(0,6;1)\Delta_{(1)}(0,6;6)}\,.
\end{aligned}
\tag{4.42}
$$

5. Conclusion

In this paper we have considered the collinear limit of multi-gluon QCD amplitudes at tree level. We have used the new MHV rules for constructing colour ordered amplitudes from MHV vertices together with the general collinear factorization formula to derive timelike splitting functions that are valid for specific numbers of negative helicity gluons with an arbitrary number of positive helicity gluons (or vice versa). In this limit, the full amplitude factorises into an MHV vertex multiplied by a multi-collinear splitting function that depends on the helicities of the collinear gluons. These splitting functions are derived directly using MHV rules. Out of the full set of MHV-diagrams contributing to the full amplitude, only the subset of MHV-diagrams which contain an internal propagator which goes on-shell in the multi-collinear limit contribute.

We find that the splitting functions can be characterised by ΔM, the difference between the number of negative helicity gluons before taking the collinear limit, and the number after. $\Delta M+1$ also coincides with the number of MHV vertices involved in the splitting functions. Our main results are splitting functions for arbitrary numbers of gluons where $\Delta M = 0,1,2$. Splitting functions where the difference in the number of positive helicity gluons $\Delta P = 0,1,2$ are obtained by the parity transformation. These general results are sufficient to describe *all* collinear limits with up to six gluons. We have given explicit results for up to four collinear gluons for all independent helicity combinations, which numerically agree with the results of ref. [64], together with new results for five and six collinear gluons. This method could be applied to higher numbers of negative helicity gluons, and via the MHV-rules for quark vertices, to the collinear limits of quarks and gluons.

We anticipate that the results presented here will be useful in developing higher order perturbative predictions for observable quantities, such as jet cross sections at the LHC.

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